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A generalization of the Bargmann–Fock representation to supersymmetry

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Abstract. In the Bargmann–Fock representation the coordinates z^i act as bosonic creation operators while the partial derivatives ∂_{z^j} act as annihilation operators on holomorphic 0-forms as states of a D-dimensional bosonic oscillator. Also considering p-forms and further geometrical objects as the exterior derivative and Lie derivatives on a holomorphic $\mathbf{C}^D$, we end up with an analogous representation for the D-dimensional supersymmetric oscillator. In particular, the supersymmetry multiplet structure of the Hilbert space corresponds to the cohomology of the exterior derivative. In addition, a 1-complex parameter group emerges naturally and contains both time evolution and a homotopy related to cohomology. The emphasis is on calculus.

1. Introduction

The conventional Bargmann–Fock representation displays the D-dimensional bosonic harmonic oscillator by using holomorphic 0-forms on a manifold $\mathbf{C}^D$ to represent states in a Hilbert space. The bosonic creation and annihilation operators are represented by z^i and ∂_{z^j}, respectively [1–3]. Thus, the Bargmann–Fock representation is a local geometrical concept. We extend this idea to include holomorphic p-forms and consider geometrical operations on these to find that the D-dimensional supersymmetric (SUSY) oscillator [4–8] is realized in every detail. This approach constitutes an alternative to the standard use of Grassmann variables, as discussed in [9, 10] and in many papers quoted therein.

One way to develop the formalism would be to reformulate the local differential geometry of the full $\mathbf{C}^D$ and impose the restriction to holomorphic quantities afterwards, since the standard geometrical apparatus is more familiar for the more general case [11, 12]. Instead, in the second section, we will develop a calculus for a purely holomorphic differential geometry and, by immediate interpretation, built up the corresponding physical system simultaneously. The only structure that is not really geometric on a holomorphic manifold but an additional ingredient is the scalar product, of which we give an alternative definition not involving any integral which would exceed the concept of holomorphic geometry. We will see that supersymmetry (SUSY) is supplied by the operator ∂. The Hamiltonian is a Lie derivative corresponding to a 1-complex-parameter group that can be split by holomorphicity into two equivalent 1-parameter groups. One can be identified with evolution in a Hilbert space. The other supplies a homotopy that occurs in the proof of Poincaré's lemma [11]. We end this section with a brief discussion of the eigenstates of two Lie derivatives, the first one being the Hamiltonian and the second one yielding coherent states.

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The concluding remarks of the third section will first comment on the underlying supergroup structure. Secondly, for completeness, we will give an integral expression for the scalar product also accomodating p-forms. This will relate our representation, which dispenses with square-integrable functions, to the coherent state representation and other familiar representations, where square-integrable functions are employed. Finally, we give a prescription that would give us the physical states represented by holomorphic forms, if we started with the geometry of the full $\mathbf{C}^D$.

In the following, all commutators are graded ones, i.e. if both entries have odd form degree, we have an anti-commutator, otherwise we have a commutator. The type of the commutator is indicated by a subscript for the convenience of the reader. The same indices in upper and lower position indicate a sum from 1 to D.

2. Holomorphic geometry and the SUSY oscillator

Consider a manifold $\mathbf{R}^{2D}$ and choose a global parametrization $x^j, y^j, j = 1, 2, \ldots, D$, where both x^j and y^j take values in $\mathbf{R}$. We combine pairs of real coordinates into $z^j = x^j + \mathrm{i}y^j$ taking values in $\mathbf{C}$, such that our manifold is now $\mathbf{C}^D$. Furthermore, we demand that any function $\mathbf{C}^D \to \mathbf{C}$ be of the form

$$f(z^i) = f_0 + f_k z^k + f_{kl} z^k z^l + \cdots \qquad f_{kl\ldots} = \text{constant} \in \mathbf{C} \tag{1}$$

i.e. holomorphic around the origin $z^1 = z^2 = \cdots = z^D = 0$. The series in equation (1) has to be convergent on all of $\mathbf{C}^D$ and we call a manifold, where such functions live $\mathbf{C}^D_{\mathrm{h}}$ (holomorphic $\mathbf{C}^D$). In fact, $f(z^i) < \text{constant} \times \exp(-\sum^D_{k=1}(z^k)^2/2)$ [3], in order to yield normalizable states.

Besides the functions, it is natural to have holomorphic vector fields on our manifold. Since we work in a fixed frame, there is a canonical decomposition

$$v = v^i \partial_{z^j} \in T\mathbf{C}^D_{\mathrm{h}} \tag{2}$$

where $v^i(z^i)$ are holomorphic functions as in equation (1) and $\partial_{z^j} = \frac{1}{2}(\partial_{x^j} - \mathrm{i}\partial_{y^j})$ are the holomorphic basis vectors $\in T\mathbf{C}^D_{\mathrm{h}}$. Among the coordinates z^i and the basis vectors ∂_{z^j} the following commutation relations hold due to $\partial_{z^j} z^i = \delta^i_j$:

$$[z^i, z^j]_- = [\partial_{z^i}, \partial_{z^j}]_- = 0 \qquad [\partial_{z^i}, z^j]_- = \delta^j_i. \tag{3}$$

This is the algebra of bosonic creators z^i and annihilators ∂_{z^j} operating on functions equation (1).

From this point of view, we can apply z^i to a 'vacuum' 1 (from the left) in order to get equation (1), which represents a general state in a bosonic Fock space. Along with the tangent space $T\mathbf{C}^D_{\mathrm{h}}$ of the holomorphic vectors v, the dual cotangent space $T^*\mathbf{C}^D_{\mathrm{h}}$, containing the holomorphic 1-forms $F^{(1)}$ that provide linear maps of the holomorphic vectors to $\mathbf{C}$, is a natural geometrical structure. Again, there is a canonical decomposition $F_j(z^i)\,\mathrm{d}z^j$ with F_j holomorphic as in equation (1) and $\mathrm{d}z^j = \mathrm{d}x^j + \mathrm{i}\,\mathrm{d}y^j$. A holomorphic p-form may be written as

$$F^{(p)}(z^i, \mathrm{d}z^j) = F_{k_1\ldots k_p}(z^i)\,\mathrm{d}z^{k_1}\ldots\mathrm{d}z^{k_p} \in \Lambda^p\mathbf{C}^D_{\mathrm{h}} \tag{4}$$

with the factors $F_{k_1\ldots k_p}(z^i)$ as in equation (1). Observe that $0 \leqslant p \leqslant D$, although the dimension of $\mathbf{C}^D_{\mathrm{h}}$ is $2D$. Finally, holomorphic forms are (finite) power series in the $\mathrm{d}z^j$

$$\Psi(z^i, \mathrm{d}z^j) = F_0(z^i) + F_k(z^i)\,\mathrm{d}z^k + F_{kl}(z^i)\,\mathrm{d}z^k\,\mathrm{d}z^l + \cdots \quad \in \Lambda\mathbf{C}^D_{\mathrm{h}} = \bigoplus_{p=0}^{D}\Lambda^p\mathbf{C}^D_{\mathrm{h}} \tag{5}$$

spanning the holomorphic exterior algebra.

On holomorphic forms the interior derivative is a natural geometrical operation that maps $\Lambda^p\mathbf{C}^D_\mathrm{h}$ to $\Lambda^{p-1}\mathbf{C}^D_\mathrm{h}$ by contraction with a holomorphic vector v. As used in [13] an interior derivative on a real manifold, induced by a real vector $u^i\partial_{x^i}$, can be written as

$$u^i\partial_{\mathrm{d}x^i} := u^i\frac{\partial}{\partial\mathrm{d}x^i} \equiv i_u \tag{6}$$

where $\partial/\partial_{\mathrm{d}x^i}$ is a Grassmann left derivative with respect to the Grassmann number $\mathrm{d}x^i$. The duality of frame and coframe is expressed by $\partial\mathrm{d}x^j/\partial\mathrm{d}x^i = \delta_i^j$. Accordingly, a vector $v = v^k(z^i)\partial_{z^k} = \frac{1}{2}v^k(z^i)\partial_{x^k} - \frac{1}{2}\mathrm{i}v^k(z^i)\partial_{y^k}$ induces an interior derivative

$$v^i\partial_{\mathrm{d}z^i} := v^i\frac{\partial}{\partial\mathrm{d}z^i} \equiv i_v \tag{7}$$

where we have defined a new Grassmann left derivative $\partial_{\mathrm{d}z^j} = \frac{1}{2}(\partial_{\mathrm{d}x^j} - \mathrm{i}\partial_{\mathrm{d}y^j})$. (On $\mathbf{C}^D$, a general vector $w = w_z^k(z^i,\bar{z}^j)\partial_{z^k} + w_{\bar{z}}^k(z^i,\bar{z}^j)\partial_{\bar{z}^k}$ induces the interior derivative $i_w \equiv w_z^k\partial_{\mathrm{d}z^k} + w_{\bar{z}}^k\partial_{\mathrm{d}\bar{z}^k}$). The duality of holomorphic frame ∂_{z^i} and holomorphic coframe $\mathrm{d}z^j$ is expressed by $\partial\mathrm{d}z^j/\partial\mathrm{d}z^i = \delta_i^j$. This gives rise to

$$[\mathrm{d}z^i,\mathrm{d}z^j]_+ = [\partial_{\mathrm{d}z^i},\partial_{\mathrm{d}z^j}]_+ = 0 \qquad [\partial_{\mathrm{d}z^i},\mathrm{d}z^j]_+ = \delta_i^j \tag{8}$$

which is the algebra of fermionic creation operators $\mathrm{d}z^j$ and annihilation operators $\partial_{\mathrm{d}z^j}$. The commutators of mixed bosonic and fermionic entries are

$$[z^i,\mathrm{d}z^j]_- = [z^i,\partial_{\mathrm{d}z^j}]_- = [\partial_{z^i},\mathrm{d}z^j]_- = [\partial_{z^i},\partial_{\mathrm{d}z^j}]_- = 0\,. \tag{9}$$

Thus, equation (4) constitutes a general state vector of a Fock space with D bosonic and D fermionic degrees of freedom.

The next task is to introduce a scalar product that makes the exterior algebra a Hilbert space. We start by defining the adjoint operation. Inspired by the algebraic properties of the elementary operators $z^i, \mathrm{d}z^j, \partial_{z^k}, \partial_{\mathrm{d}z^l}$, we define their adjoints by

$$(z^i)^+ := \partial_{z^i} \qquad (\mathrm{d}z^j)^+ := \partial_{\mathrm{d}z^j} \tag{10}$$

with the rules $(A+B)^+ = A^+ + B^+$, $(AB)^+ = B^+A^+$, $(A^+)^+ = A$ and $c^+ = c^*$, if $c = \text{constant} \in \mathbf{C}$. While a state vector $\Psi(z^i,\mathrm{d}z^j)$ is a power series in $z^i,\mathrm{d}z^j$, its dual $\Psi^+(\partial_{\mathrm{d}z^l},\partial_{z^k})$ is a power series in $\partial_{z^k},\partial_{\mathrm{d}z^l}$. The prescription for the scalar product of two state vectors Ψ and Ξ from the exterior algebra is the following:

$$\langle\Psi(z^i,\mathrm{d}z^j)|\Xi(z^k,\mathrm{d}z^l)\rangle := \Psi^+(\partial_{\mathrm{d}z^j},\partial_{z^i})\Xi(z^k,\mathrm{d}z^l)|_{z^1=z^2=\cdots z^D=\mathrm{d}z^1=\mathrm{d}z^2=\cdots=\mathrm{d}z^D=0} \tag{11}$$

i.e. perform all the derivations in Ψ^+ on Ξ and put the remaining factors $z^i,\mathrm{d}z^j$ to zero.

Up to now, we implicitly used the exterior derivative d that maps bosonic z^j to fermionic $\mathrm{d}z^j$. The exterior derivative may be decomposed as $\mathrm{d} = \partial + \bar\partial = \mathrm{d}z^i\,\partial_{z^i} + \mathrm{d}\bar{z}^i\,\partial_{\bar{z}^i}$. But since $\bar\partial = 0$ on $\mathbf{C}^D_\mathrm{h}$, d reduces to $\partial = \mathrm{d}z^i\,\partial_{z^i}$, which is nilpotent $\partial^2 = 0$. Its adjoint is $\partial^+ = z^i\partial_{\mathrm{d}z^i}$ mapping fermionic $\mathrm{d}z^j$ to bosonic z^j, being also nilpotent $(\partial^+)^2 = 0$. ∂^+ is an interior derivative with respect to the vector field $z^i\partial_{z^i}$.

The final prominent geometrical object that we consider is the holomorphic Lie derivative with respect to a vector field v, which is an anti-commutator

$$\mathcal{L}_v = [\partial, v^i\partial_{\mathrm{d}z^i}]_+\,. \tag{12}$$

(On $\mathbf{C}^D$, by definition $L_w = [\partial+\bar\partial, w_z^i\partial_{\mathrm{d}z^i} + w_{\bar{z}}^i\partial_{\mathrm{d}\bar{z}^i}]_+$ corresponds to a 1-complex-parameter group. There is a decomposition $L_w = \mathcal{L}_w + \bar{\mathcal{L}}_w := [\partial, w^i\partial_{\mathrm{d}z^i} + w_{\bar{z}}^i\partial_{\mathrm{d}\bar{z}^i}]_+ + [\bar\partial, w_z^i\partial_{\mathrm{d}z^i} + w_{\bar{z}}^i\partial_{\mathrm{d}\bar{z}^i}]_+$, where each $\mathcal{L}_w$ and $\bar{\mathcal{L}}_w$ corresponds to a 1-complex-parameter group.) From equation (12), an important property follows immediately,

$$[\mathcal{L}_v, \partial]_- = 0\,. \tag{13}$$

The Lie derivative is self-adjoint if the interior derivative is adjoint to ∂. This is true for

$$H := \mathcal{L}_{z^i \partial_{z^i}} = [\partial, \partial^+]_+ = z^i \partial_{z^i} + \mathrm{d}z^i\, \partial_{\mathrm{d}z^i}\,. \tag{14}$$

The first term counts the powers in the coordinates of an expression, that it is applied on. So we define the boson number operator $N := z^i \partial_{z^i}$, of form degree 0. The second term counts the form degree if it is applied to a p-form. Accordingly, we define the fermion number operator $P := \mathrm{d}z^i\, \partial_{\mathrm{d}z^i}$, which also has form degree 0. Due to equation (13), ∂ and ∂^+ are conserved

$$[\partial, H]_- = [\partial^+, H]_- = 0\,. \tag{15}$$

Equations (14) and (15) represent the algebra of a D-dimensional SUSY oscillator with the SUSY Hamiltonian H and charges ∂ and ∂^+, which can be combined into two self-adjoint charges $Q_1 = \partial + \partial^+$ and $Q_2 = -\mathrm{i}(\partial - \partial^+)$.

Locally a holomorphic Lie derivative is just a total derivative with respect to a complex parameter $\theta = \tau + \mathrm{i}t$. Since for our trivial topology local and global concepts coincide, we have on the entire space

$$-\frac{\mathrm{d}}{\mathrm{d}\theta}\Psi(\theta) = H\Psi(\theta) \tag{16}$$

for any

$$\Psi(\theta) = \Psi(z^i(\theta), \mathrm{d}z^j(\theta)) = \Psi(z^i \mathrm{e}^{-\theta}, \mathrm{d}z^j\, \mathrm{e}^{-\theta}) = \mathrm{e}^{-\theta H}\Psi(z^i, \mathrm{d}z^j) \in \Lambda\mathbf{C}_\mathrm{h}^D \tag{17}$$

where $z^i \equiv z^i(0)$ and $\mathrm{d}z^j \equiv \mathrm{d}z^j(0)$.

Hence, holomorphic forms are subject to an 'evolution' in the parameter θ corresponding to a trivial line bundle of charts $\mathbf{C}(\theta) \times \mathbf{C}_\mathrm{h}^D(z^1, \dots, z^D)$. θ parametrizes a sequence of charts on $\mathbf{C}_\mathrm{h}^D$ each representing the system at an 'instant' θ. Although each chart is a Hilbert space, the whole line bundle $\mathbf{C}(\theta) \times \mathbf{C}_\mathrm{h}^D(z)$, representing a complex 'evolution', is not a Hilbert space.

It is, however, possible to use inherent information to eliminate the Euclidean τ such that the remaining bundle is a Hilbert space as we are accustomed to. Along with equation (16)

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}}\Psi(\theta) = 0 \tag{18}$$

also holds, since $\Psi(z^i \mathrm{e}^{-\theta}, \mathrm{d}z^j\, \mathrm{e}^{-\theta})$ is holomorphic in θ. We use equation (18) to eliminate τ from (16)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = H\Psi(t)\,. \tag{19}$$

This identifies t as the time and the system evolves within a bundle of charts $U(1)(t) \times \mathbf{C}_\mathrm{h}^D(z^1, \dots, z^D)$, which is a Hilbert space as a whole and the scalar product is preserved in t-evolution automatically.

Conversely, we can eliminate t and end up with

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\Psi(\tau) = H\Psi(\tau) \tag{20}$$

corresponding to an alternative bundle $\mathbf{R}_+(\tau) \times \mathbf{C}_\mathrm{h}^D(z^1, \dots, z^D)$. τ-evolution preserves the scalar product due to $\tau^+ = -\tau$, but the bundle itself is not a Hilbert space, cf [13]. Considering the limit $\tau \to \infty$ of equation (17), by use of (14), we prove that any state with $p \geqslant 1$ is ∂-exact, if and only if it is ∂-closed, which is Poincaré's lemma on C_h^D. More precisely, we find that the only non-trivial cohomology class contains the constant numbers with zero eigenvalue of H, representing the normalized state 1, which is the only state left in the limit $\tau \to \infty$. Thus it follows that, with the exception of the SUSY singlet vacuum 1, all other states are paired by the operator ∂. This conforms, of course, with the fact that the conditions $\partial\Psi = 0$, while $\Psi \neq \partial \ldots$, alternatively expressed by $\partial\Psi = \partial^{+}\Psi = 0$, are solved directly by $\Psi =$ constant

We emphasize that (19) and (20) on $\mathbf{C}_{\mathrm{h}}^{D}$ are equivalent. The first one describes evolution on a Hilbert space, the second one encodes the cohomology of the underlying manifold by supplying a homotopy of the manifold, which renders the non-trivial cohomology classes in the limit $\tau \to \infty$. This makes transparent the intimate relation between the SUSY and the time evolution which is a characteristic feature of any SUSY theory.

Since the holomorphic Lie derivative is an operator of form degree zero, it commutes with the fermion number operator P, of which the eigenstates are homogeneous p-forms corresponding to the eigenvalue p. Hence the eigenstates of a holomorphic Lie derivative can always be arranged to yield homogeneous p-forms. The ∂-operator provides a pairing by mapping a non-closed p-form to an exact $(p+1)$-form, which both span a two-dimensional eigenspace corresponding to an eigenvalue of $\mathcal{L}_v$ (up to further degeneracy not related to SUSY). Two Lie derivatives are particularly interesting:

(i) The Hamiltonian H yielding a complete set of eigenstates given by monomials

$$\begin{aligned}\Phi_E^{(p)} &= \frac{1}{\sqrt{n_1!n_2!\ldots n_D!}}(z^1)^{n_1}(z^2)^{n_2}\ldots(z^D)^{n_D}(\mathrm{d}z^1)^{p_1}(\mathrm{d}z^2)^{p_2}\ldots(\mathrm{d}z^D)^{p_D}\\ n_j &= 0,1,2,\ldots \qquad p_j = 0,1 \qquad p = p_1+p_2+\cdots+p_D\end{aligned} \tag{21}$$

corresponding to the energy value $E = n_1+n_2+\cdots+n_D+p_1+p_2+\cdots+p_D$ and being orthonormal

$$\langle\Phi_E^{(p)}, \Phi_{E'}^{(p')}\rangle = \delta_{n_1n_1'}\delta_{n_2n_2'}\ldots\delta_{n_Dn_D'}\delta_{p_1p_1'}\delta_{p_2p_2'}\ldots\delta_{p_Dp_D'}\,. \tag{22}$$

(ii) The Lie derivative corresponding to rigid translations, which reduces to a simple directional derivative on any form

$$\mathcal{L}_{c^i\partial_{z^i}} = [\partial, c^i\partial_{\mathrm{d}z^i}]_+ = c^i\partial_{z^i} \qquad c^i = \text{constant} \in \mathbf{C}. \tag{23}$$

The eigenvalue problem reads

$$c^i\partial_{z^i}\kappa = \alpha\kappa \qquad \alpha = c^i\alpha_i \in \mathbf{C} \tag{24}$$

and its solution is

$$\kappa_\alpha^{(p)} = \mathrm{e}^{-\frac{1}{2}\alpha^{*i}\alpha_i}\mathrm{e}^{\alpha_i z^i}(\mathrm{d}z^1)^{p_1}(\mathrm{d}z^2)^{p_2}\ldots(\mathrm{d}z^D)^{p_D} \tag{25}$$

after normalization, using $\exp(\alpha^{*i}\partial_{z^i})\Psi(z^j, \mathrm{d}z^k) = \Psi(z^j+\alpha^{*j}, \mathrm{d}z^k)$. Coherent states of different form degree are orthogonal, but the scalar product of two arbitrary coherent states is

$$\langle\kappa_\alpha^{(p)}, \kappa_{\alpha'}^{(p')}\rangle = \mathrm{e}^{-\frac{1}{2}(\alpha^{*i}\alpha_i+\alpha^{*\prime i}\alpha_i'-2\alpha^{*i}\alpha_i')}\delta_{p_1p_1'}\delta_{p_2p_2'}\ldots\delta_{p_Dp_D'}\,. \tag{26}$$

The $\kappa_\alpha^{(p)}$ generalize the coherent states of the bosonic harmonic oscillator, which are contained for the special case $p = 0$. The characteristic properties of the bosonic coherent states are preserved for arbitrary p. In particular, they constitute an overcomplete set, if the α_i are not restricted to a subset of the $\mathbf{C}^D$ plane, which is just complete [14] and they are minimum uncertainty states with respect to the position operators $(1/\sqrt{2})(z^i+\partial_{z^i})$ and momentum operators $(\mathrm{i}/\sqrt{2})(z^i-\partial_{z^i})$ for any p. The coherent states above have to be distinguished from 'supercoherent states' as discussed in [15], which employ Grassmann parameters.

3. Concluding remarks

The whole formalism is invariant under $U(D)$ transformations $z^i \to z'^i = \Lambda^i{}_j z^j$ where $\Lambda^i{}_j \in U(D)$, which is the symmetry group of the classical bosonic oscillator. In fact, as is discussed in [7], the full symmetry group of the SUSY oscillator is the supergroup $U(D/D)$, which combines the $U(D)$ transformation with the interchange of objects paired by the SUSY charges. The Hamiltonian generates an Abelian subgroup corresponding to evolution $U(D/D) = U(1)(t) \times SU(D/D)$. In contrast the strictly real approach of [13] has only $O(D)$ in the bosonic sector being promoted to $O(D/D)$, which contains no continuous Abelian subgroup that could account for evolution.

So far, we have neither used any non-holomorphic quantities, nor did we use a metric on the manifold. By means of these additional ingredients, it is possible to give an integral version of the scalar product equation (11) for a p- and a q-form by using the Hodge star

$$\langle \Psi^{(p)}(z^i, \mathrm{d}z^j) | \Xi^{(q)}(z^k, \mathrm{d}z^l) \rangle = \frac{1}{2^p \pi^D} \int_{\mathbf{C}^D} \mathrm{e}^{-z^m \bar{z}_m} \Psi^{(p)}(z^i, \mathrm{d}z^j) * \bar{\Xi}^{(q)}(\bar{z}^k, \mathrm{d}\bar{z}^l) \tag{27}$$

where the Hodge star is with respect to the standard Euclidean metric and in our coordinates the corresponding orthonormal frame is $\{\partial_{x^1}, \partial_{y^1}, \partial_{x^2}, \partial_{y^2}, \ldots, \partial_{x^D}, \partial_{y^D}\}$. Absorbing the exponential into the entries of the scalar product, we have

$$\langle \Psi^{(p)} | \Xi^{(q)} \rangle = \frac{1}{2^p \pi^D} \int_{\mathbf{C}^D} \langle \Psi^{(p)} | \bar{z}^i, \mathrm{d}\bar{z}^j \rangle * \langle \bar{z}^k, \mathrm{d}\bar{z}^l | \Xi^{(q)} \rangle\,. \tag{28}$$

The scalar product vanishes whenever $p \neq q$, because for $p - q > 0$ a $(p + 2D - q)$-form vanishes by antisymmetry and for $p - q < 0$ we integrate over a set of measure zero in $2D$-dimensional space. For arbitrary elements of the exterior algebra, equations (27) or (28) have to be applied after decomposition into homogeneous p-forms. For 0-forms the above version for the scalar product reproduces the usual Bargmann–Fock prescription [1, 2] and coincides with the coherent state representation for the bosonic harmonic oscillator [14]. It is remarkable that we have managed to integrate fermionic quantities with conventional integration, such that our integral version of the scalar product is different from the usual Grassmann integration à la Berezin [16].

Finally, if we had developed the formalism on the entire $\mathbf{C}^D$, thus considering forms $\Gamma(z^i, \mathrm{d}z^j, \bar{z}^k, \mathrm{d}\bar{z}^l)$, we would have needed a prescription in order to select the holomorphic forms that represent the physical states. This is accomplished by imposing $\bar{\partial}\Gamma = 0$, while $\Gamma \neq \bar{\partial} \ldots$. Therefore $\Lambda \mathbf{C}^D_{\mathrm{h}} = \bigoplus_{p=0}^{D} \Lambda^p \mathbf{C}^D_{\mathrm{h}} \equiv \mathcal{H}(\mathbf{C}^D, \bar{\partial}) = \bigoplus_{p=0}^{D} \mathcal{H}^p(\mathbf{C}^D, \bar{\partial})$ such that we are actually working in the non-trivial cohomology sector of $\bar{\partial}$.

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